18
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A short history of model theory

18.1 'A new branch of metamathematics'

In 1954, Alfred Tarski wrote:

> Within the last years a new branch of metamathematics has been developing. It is called the *theory of models* and can be regarded as a part of the semantics of formalized theories. The problems studied in the theory of models concern mutual relations between sentences of formalized theories and mathematical systems in which these sentences hold.[1]

In these words Tarski defined and named a new branch of mathematics, which today we know as *mathematical model theory*, or simply as *model theory*. The present essay will trace some of the main themes in the history of mathematical model theory, roughly up to the beginning of the twenty-first century. (What would non-mathematical model theory be? One example—there are others—is the 'model-theoretic syntax' developed by the linguists Pullum and Scholz;[2] it has historic links with mathematical model theory.)

Although Tarski named the new subject, he certainly didn't own it. Already before 1954 Anatoliĭ Mal'tsev and Abraham Robinson had published results that became as characteristic of the subject as any of Tarski's own contributions to it; we will come to their work below. Tarski's main role—apart from collecting a stellar group of young researchers around him in Berkeley and giving them problems to work on—had been to take up some earlier questions from the heuristic fringes of mathematics, and show how to give them mathematical precision.

Tarski refers to 'mathematical systems'. He means what we now usually call *structures*—they have a domain of elements, and a collection of relations, functions, and distinguished elements defined in this domain and named by specified relation symbols, function symbols and individual constants. Structures in this sense are an invention of the second half of the nineteenth century—for example David Hilbert handled them freely in his *Grundlagen der Geometrie*.[3] A system is a collection of things brought together in an orderly way. For Hilbert and his German predecessors, it seems that the things brought together were the elements of the structure. Thus, Richard Dedekind used the name 'System' both for structures and for

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[1] Tarski (1954: 572). [2] Pullum and Scholz (2001). [3] Hilbert (1899).

sets—apparently he thought of a structure as a set that comes with added features.[4] Heinrich Weber and Hilbert spoke of 'Systeme von Dingen' [systems of things], to distinguish from axiom systems. On the other hand George Boole,[5] adapting the language of George Peacock,[6] had spoken of a 'system of interpretation'; for Boole, the things brought together were the operations as interpretations of symbols, for example $+$ and $\times$ as function symbols and 0 as individual constant. Logicians regarded the interpretation of symbols by relations etc. of the structure as central, so one finds structures being referred to as 'interpretations' well into the twentieth century.

In model theory the name 'system' persisted until it was replaced by 'structure' in the late 1950s, it seems under the influence of Robinson and Bourbaki.[7]

Tarski speaks of 'sentences'. Mostly these were taken as concatenated strings of formal symbols. But already in the 1930s, Kurt Gödel was handling languages of uncountable cardinality, with arbitrary objects as symbols, using any suitable functions to replace concatenation of symbols.[8] Mal'tsev did likewise.[9] By the 1950s it was taken for granted that a 'sentence' could be a purely set-theoretic object.

Tarski also refers to the notion of a sentence 'holding in' a structure. The notion of a statement 'holding in' some contexts and not others is not a particularly mathematical one; for example a legal journal of 1900 speaks of 'contravening the rule held in the above cases'. It was one of a number of idioms that mathematicians had used to express what we now mean by saying that a structure is a *model of*, or *satisfies*, a formal sentence. Alessandro Padoa spoke of a structure 'verifying' axioms.[10] The word 'satisfy' in this context may be due to Edward V. Huntington;[11] Huntington was a member of the group of American mathematicians around Eliakim H. Moore and Oswald Veblen who, in the early twentieth century, made a systematic study of axiomatically defined classes of structures.[12] We can trace back the use of the word 'model' itself to the seventeenth century geometers who spoke of gypsum or paper 'models' of geometrical axioms. The term 'model' for abstract structures appeared during the 1920s in writings of the Hilbert school.[13]

18.2 Replacing the old metamathematics

One feature of the early work on models of axioms was the looseness of some of the formulations. Three examples follow. In each of them an informal method was in use around 1900, then Tarski attempted a non-model-theoretic formalisation in

[4] Dirichlet and Dedekind (1871) and Dedekind (1872). [5] Boole (1847: 3). [6] Peacock (1833). [7] A. Robinson (1952) and Bourbaki (1951). [8] Gödel (1932). [9] Mal'tsev (1936). [10] Padoa (1900). [11] For example in Huntington (1902). [12] Scanlan (2003). [13] von Neumann (1925) and Fraenkel (1928: 342). R. Müller (2009) gives historical information on the use of the word 'model' in model theory and elsewhere.

the 1930s, and finally in the 1950s a model-theoretic formalisation was given which is now widely regarded as canonical.

Categoricity

Veblen introduced the notion of categoricity:

> [...] a system of axioms is categorical if it is sufficient for the complete *determination* of a class of objects or elements.[14]

to which he added a brief informal explanation of isomorphisms. [*See §7.2 footnote 3.*] Veblen's word 'sufficient' harks back to Huntington's paper,[15] where a set of postulates (i.e. axioms) is said to be 'sufficient' if 'there is essentially *only one*' structure that satisfies the postulates. In 1935, Tarski attempted to tidy up the notion of categoricity as follows.[16] First he assumed that the system of axioms is finite, so that its conjunction can be written as a single formula of an appropriate higher-order logic

$$a(x, y, z, \ldots)$$

where the variables 'x' etc. represent the non-logical notions in the axioms (for example 'point', 'line'). Then he wrote

$$R\frac{(x'\,, y'\,, z'\,, \ldots)}{(x'', y'', z'', \ldots)}$$

for the formal statement that R is a permutation of the universe of individuals, which takes x' to x'', y' to y'' etc. Finally he defined the axiom system $a(x, y, z, \ldots)$ to be categorical if the higher-order statement

$$\forall x' \forall y' \forall z' \ldots \forall x'' \forall y'' \forall z'' \ldots \\ \left(a(x', y', z', \ldots) \wedge a(x'', y'', z'', \ldots) \rightarrow \exists R\, R\frac{(x'\,, y'\,, z'\,, \ldots)}{(x'', y'', z'', \ldots)} \right)$$

is 'logically provable'. Note that at this date, Tarski's notion of 'categorical' made no use of the notion of an axiom 'holding in' a structure. In short, it was not model-theoretic. Nor was it objective, since the notion of 'logically provable' in higher-order logic depends on what axioms you accept for this logic.

By the early 1950s, all the definitions were in place to allow the definition that a theory T is categorical if and only if T has exactly one model up to isomorphism. [*See §7.2.*] But by the 1950s the preferred logical language had become first-order logic, and the Upward Löwenheim–Skolem Theorem implied that no first-order theory with infinite models is categorical. Accordingly Vaught defined a theory T

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[14] Veblen (1904: 347). [15] Huntington (1902). [16] Tarski (1935a).

to be λ-*categorical* (for a cardinal λ) if T has, up to isomorphism, exactly one model of cardinality λ.[17] [*See §17.3.*] (The cardinality of a structure is that of its domain of elements.) We will see below how this became one of the most fertile definitions in model theory.

Padoa's method

Padoa proposed a criterion for showing that in the context of an axiomatic theory T, no formal definition of a notion A in terms of notions $B_1, \ldots, B_n$ can be deduced from the axioms T.[18] The criterion was that there exist two interpretations of T which agree in how they interpret $B_1, \ldots, B_n$ but disagree in the interpretation of A. Padoa sketched proofs of the necessity and sufficiency of this criterion. But today it is obvious that he couldn't hope to prove necessity without saying more about how he understood 'deducible from T'; and in fact his proof of necessity is just a blurred repetition of his proof of sufficiency. Today 'Padoa's method' is generally taken to consist of a model-theoretic criterion for a syntactic notion. But Tarski's reformulation of Padoa's proposal removed all model-theoretic notions and translated Padoa's proposal into pure syntax.[19]

Padoa's method had a bumpy ride into the new context of model theory. In 1953, Evert Beth proved that Padoa's claim was true at least for first-order logic.[20] Beth took Padoa's criterion model-theoretically. But since at this date there was no clear model-theoretic route from the absence of a definition to the truth of the criterion, Beth translated the criterion into proof theory along Tarski's lines but within first-order logic, and then used his own adaptation of Gentzen's cut-free proofs to build the required models. Tarski, through his student Solomon Feferman,[21] responded that, since Beth's Theorem was proof-theoretic, it would be best to play down the model-theoretic form of the criterion, which was only incidental to the main result. Soon afterwards another member of the Berkeley group, William Craig, reworked Beth's use of cut-free proofs, and thereby discovered the Craig Interpolation Theorem.[22] Almost at once it came to notice that Abraham Robinson in Toronto had already proved a model-theoretic result equivalent to the Interpolation Theorem, using purely model-theoretic methods.[23] From this point onwards it was accepted that model theory and proof theory could each feed useful information to the other. In particular, Feferman proved a number of model-theoretic results by giving proof-theoretic demonstrations of a range of interpolation theorems.[24]

[17] Vaught (1954). [18] Padoa (1900). [19] Tarski (1935a). [20] Beth (1953). [21] van Ulsen (2000: 138). [22] Craig (1957a,b). [23] A. Robinson (1956a). [24] For example in Feferman (1974).

Proofs of logical independence

Padoa (1900) related his proposal to another heuristic that was already in use. Namely, we can show that a formal axiom ψ doesn't follow from formal axioms $\varphi_1, \dots, \varphi_n$ by exhibiting an interpretation of the symbols in these axioms, which makes $\varphi_1, \dots, \varphi_n$ hold but ψ fail to hold. This method had been used by Felix Klein and Eugenio Beltrami to show that Euclid's parallel postulate doesn't follow from his other axioms. In the years around 1900, Giuseppe Peano, Hilbert, and Huntington all applied the method.[25]

Gottlob Frege took umbrage at Hilbert's use of this method. One assumption that Hilbert made was that the non-logical symbols in the axioms are ambiguous in the sense that they can be interpreted in different ways in different structures, even within the same mathematical discourse. Frege commented:

> In der Tat, wenn es sich darum handelte, sich und andere zu täuschen, so gäbe es kein besseres Mittle dazu, als vieldeutige Zeichen. [Indeed, if it were a matter of deceiving oneself and others, there would be no better means than ambiguous signs.][26]

Frege's comments were not all negative. He went on to sketch a way in which Hilbert's arguments could be brought into a formal deductive system, by replacing the 'ambiguous signs' by higher-order variables and then proving formal statements that quantified universally over these variables, very much as in Tarski's later work of the 1930s.[27]

In this case it will be best to jump straight to the 1950s to see how Frege's concerns were answered within model theory. A paper of Tarski and Vaught indicates how to write within pure set theory a recursive definition of the relation:[28]

$$\text{Sentence } \varphi \text{ is true in structure } \mathcal{M}. \tag{1}$$

Standard methods allow this recursive definition to be reduced to a set-theoretic formula $\theta(\mathcal{M}, \varphi)$. The independence notion mentioned by Padoa above can then be formalised in pure set theory as

$$\exists \mathcal{M}(\theta(\mathcal{M}, \varphi_1) \wedge \dots \wedge \theta(\mathcal{M}, \varphi_n) \wedge \neg\theta(\mathcal{M}, \psi)).$$

Hilbert's independence proofs in his *Grundlagen der Geometrie* can be read as proving set-theoretic sentences of this form,[29] and it then becomes a standard but tedious exercise to translate Hilbert's proofs into purely set-theoretic arguments. In these resulting arguments there is no mention of the meanings of symbols, since 'meaning' is not a set-theoretic notion. Thus Frege's complaint about ambiguous signs is met. (Tarski and Vaught use first-order logic, and some of Hilbert's formulations were not first-order; but set-theoretic formulas corresponding to θ can be

[25] Peano (1891), Hilbert (1899), and Huntington (1902). [26] Frege (1906: 307). [27] Frege (1906) and Tarski (1935a). [28] Tarski and Vaught (1958). [29] Hilbert (1899).

found for any other reasonable logic.) Frege had other objections, for example to Hilbert's use of the word 'axiom'. But 'axiom' is not a set-theoretic notion either, so this and all similar objections lose their purchase.

Now we can go back to the 1930s to see where the formula $\theta(\mathcal{M}, \varphi)$ came from. In 1933 Tarski published a paper in which he considered any formalised theory T satisfying certain conditions;[30] one of the conditions was that the symbols of T have fixed and known meanings, in such a way that every sentence of T is either true or false. This included the case where $\mathcal{M}$ is a fixed structure and φ is a formal sentence whose non-logical symbols are interpreted as in $\mathcal{M}$. He showed how to construct a metamathematical formula θ', using only higher order logic, syntax and symbols expressing the notions expressible by symbols of T, such that $\theta'(\varphi)$ is true if and only if φ is a true sentence of T. [*See §§1.3, 12.4, 12.A.*]

Tarski's famous 'Concept of Truth' paper is a translation of the expanded German version of his 1933 paper.[31] None of these versions of the paper should be counted as model-theoretic; in fact neither the word 'model' nor any equivalent expression occurs in any of them. But Tarski wanted to show that his truth definition could be used to give a precise and rigorously defined meaning to the relation (1) with $\mathcal{M}$ variable. Here he ran up against the problem that had vexed Frege. Namely, how do we deal with the notion of giving a meaning in $\mathcal{M}$ to a symbol in φ which might already have another meaning? Tarski came to suppy an answer remarkably close to Frege's.[32] Namely, he replaces the non-logical symbols in φ by variables $\bar{x}$, and then uses his truth definition to express that $\mathcal{M}$ satisfies the resulting formula $\varphi_0(\bar{x})$. From the later point of view of model theory, this procedure carries irrelevant clutter. But it can be converted into a formula $\theta(\mathcal{M}, \varphi)$ expressing (1), in set theory or some suitable higher order logic.

Tarski in the 1950s had a clean mathematical definition of (1), but he still tended to avoid the use of any notation such as $\mathrm{Mod}(T)$ for the class of models of the theory T.[33] If one also writes K for the set of sentences true in all the structures of the class K, then there are certain fundamental facts that we expect to see set down, for example

$$T \subseteq \mathrm{Th}(K) \text{ iff } \mathrm{Mod}(T) \supseteq K$$

But this group of facts are found in Abraham Robinson's doctoral thesis of 1949,[34] not in Tarski's model-theoretic papers.

[30] Tarski (1933). [31] Tarski (1983: Paper VIII). [32] Tarski (1936, 1994); followed by Tarski's student Andrzej Mostowski in his (1948). [33] For T a single sentence this notion does appear briefly in Definition 14(ii) on p.710 of his 1952. [34] A. Robinson (1951: 36–7).

18.3 Definable relations in one structure

The method of quantifier elimination

In his 'Concept of truth' paper, Tarski presents several examples of truth definitions for different kinds of language. He describes one of them as 'purely accidental'.[35] In this example he considers what today we would call the structure $\mathcal{M}$ of all subsets of a given set a, with relation $\subseteq$; he discusses what can be said about $\mathcal{M}$ using the corresponding first-order language L. (This may be an anachronism; one could also describe his example as the structure consisting of the set a with no relations, and a corresponding monadic second-order language.) Tarski works out an explicit definition of the relation 'φ is true in $\mathcal{M}$', where φ ranges over the sentences of L.

This truth definition might be accidental, but Tarski's decision to mention it was not. Leopold Löwenheim had already studied the same example within the context of the Peirce–Schröder calculus of relatives, and he had proved a very suggestive result.[36] In modern terms, Löwenheim had shown that there is a set of 'basic' formulas of the language L with the property that every formula φ of L can be reduced to a Boolean combination ψ of basic formulas which is equivalent to φ in the sense that exactly the same assignments to variables satisfy it in $\mathcal{M}$. Thoralf Skolem and Heinrich Behmann had reworked Löwenheim's argument so as to replace the calculus of relatives by more modern logical languages.[37] In 1927, Cooper H. Langford applied the same ideas to dense or discrete linear orderings.[38]

Tarski realised that not only the arguments of Löwenheim and Skolem, but also the heuristics behind them, provided a general method for analysing structures. This method became known as the *method of quantifier elimination*. In his Warsaw seminar, starting in 1927, Tarski and his students applied it to a wide range of interesting structures. An important example was the ordered abelian group of integers—not the natural numbers—with symbols for 0, 1, $+$ and $<$.[39] Another was the ordered field of real numbers.[40] In both these cases the method yielded (a) a small and easily described set of basic formulas, (b) a description of all the relations definable in the structure by first-order formulas, (c) an axiomatisation of the set of all first-order sentences true in the structure, and (d) an algorithm for testing the truth of any sentence in the structure. (Here (b) comes at once from (a). For (c), one would write down any axioms needed to reduce all formulas to Boolean combinations of basic formulas, and all axioms needed to determine the truth or falsehood of basic sentences. Then (d) follows since the procedure for reducing to basic formulas is effective.)

In principle the method of quantifier elimination tells us, for any structure M, what are the sets and relations on the domain of $\mathcal{M}$ that are definable by formulas

[35] Tarski (1933: §3). [36] Löwenheim (1915: §4). [37] Skolem (1919: §4) and Behmann (1922).
[38] Langford (1926/27a,b). [39] Presburger (1930) and supplement. [40] Tarski (1931).

of the first-order language appropriate for $\mathcal{M}$. In practice we may lack the skill or the information needed to carry the method to a conclusion. But thanks to earlier work using this method, model theorists in the 1950s had at their disposal a large amount of information about the first-order definable relations in various important mathematical structures. This certainly helped to make the definable relations of a structure one of the fundamental tools of model theory. (In 1910, Hermann Weyl had introduced the class of first-order definable relations of a relational structure, but without using a formal language.)[41]

In some cases, but not all, the method showed that every definable relation in the structure is defined by a quantifier-free formula. Joseph Shoenfield, in his textbook,[42] said that a theory T *admits elimination of quantifiers* if every formula of the language of T is equivalent, provably in T, to a quantifier-free formula. He gave a model-theoretic sufficient condition for a first-order theory to admit elimination of quantifiers, and showed that some of the results of the method of quantifier elimination could be recovered easily by using this condition. Soon afterwards, necessary and sufficient model-theoretic conditions for admitting quantifier elimination were found.[43]

For most model theorists, these new methods won hands down against the sometimes heavy syntactic calculations that were needed for the method of quantifier elimination. Tarski dissented. As late as 1978 he was defending the method of quantifier elimination against modern methods

> [...] which often prove more efficient. [...] It seems to us that the elimination of quantifiers, whenever it is applicable to a theory, provides us with direct and clear insight into both the syntactical structure and the semantical contents of that theory—indeed, a more direct and clearer insight than the modern more powerful methods to which we referred above.[44]

The method of quantifier elimination works on just one structure at a time. It involves no comparison of structures. For example Tarski applied it to the ordered field of reals, and discovered among other things that the sets of reals definable in this field by first-order formulas are precisely the unions of finitely many sets, each of which is either a singleton or an open interval with endpoints either in the field or $\pm\infty$. Ordered structures with this property are said to be *o-minimal*, following Anand Pillay and Charles Steinhorn.[45] [*See §4.10, Definition 4.19.*] Tarski also found a set T of sentences which axiomatises the field, in the sense that a first-order sentence is true in the field if and only if it is provable from T. It was realised some time later that T is precisely the set of axioms defining real-closed fields. From the calculations in the quantifier elimination, it then followed at once that every real-closed field is o-minimal. So Tarski proved a theorem about a class of structures, but the

[41] Weyl (1910). [42] Shoenfield (1967: 83). [43] For example Feferman (1968: 81–2).
[44] Doner et al. (1978: 1–2). [45] Pillay and Steinhorn (1984).

theorem was proved by a procedure that applied separately to each structure in the class. There was never any direct comparison of structures.

In fact Tarski's quantifier elimination for the reals had much wider ramifications even than this. Lou van den Dries had pointed out in 1984 that the o-minimality of the field of real numbers already gives strong information about definable relations of higher arity—in particular, it allows one to recover the cell decomposition of semialgebraic sets in real geometry.[46] Julia Knight, Pillay, and Steinhorn generalised this cell decomposition to all o-minimal structures, and showed that any structure elementarily equivalent to an o-minimal structure is also o-minimal.[47] O-minimal structures became one of the most productive tools for applications of model theory, thanks largely to the insightful enthusiasm of van den Dries and some deep applications by Alex Wilkie.[48]

In 1959, Feferman and Vaught published a paper in which they study a structure $\mathcal{M}$ of the following form.[49] An indexed family $(\mathcal{N}_i : i \in I)$ of structures is given, and $\mathcal{M}$ is the Cartesian product. [*See §13.c for notation.*] They apply the method of quantifier elimination to $\mathcal{M}$, but with a twist: instead of showing that each formula $\varphi(\bar{x})$ is equivalent to a Boolean combination of basic formulas, they find for each formula $\varphi(\bar{x})$ a formula Φ in the language of the powerset Boolean algebra $\wp(I)$, and formulas $\theta_1(\bar{x}), \ldots, \theta_n(\bar{x})$ such that, writing $X_k(\bar{a})$ for the set of indices $i \in I$ such that the projection of $\bar{a}$ to $\mathcal{N}_i$ satisfies $\theta_i(\bar{x})$ in $\mathcal{N}_i$, the statement

$$\bar{a} \text{ satisfies } \varphi(\bar{x}) \text{ in } \mathcal{M}$$

holds if and only if

$$(X_1(\bar{a}), \ldots, X_n(\bar{a})) \text{ satisfies } \Phi \text{ in } \wp(I).$$

Having got this far, they were able to prove analogous theorems for various other constructions besides Cartesian product. (The list has been expanded since.)[50] The mind boggles at how these results could ever have been discovered. In fact we know the history, and an important ancestor of the results is work of Mostowski,[51] applying a form of quantifier elimination to show, for example, that the set of sentences true in an initial ordinal with the operation of natural addition of ordinals is a decidable set.

Before we leave the topic of quantifier elimination, we should note a quantifier elimination given by Angus Macintyre for p-adic number fields in a suitable first-order language.[52] Macintyre's reduction of the definable sets to Boolean combinations

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[46] van den Dries (1984). [47] J. F. Knight et al. (1986). [48] van den Dries (1998) and, for example, Wilkie (1996). [49] Feferman and Vaught (1959). [50] Makowsky (2004). [51] Mostowski (1952). [52] Macintyre (1976).

of basic sets was exactly what Jan Denef needed in order to evaluate certain p-adic integrals.[53] (This marriage of quantifier elimination and integration was soon extended to other cases.) One of Macintyre's concerns throughout his career has been to use first-order logic in order to bring mathematical notions into tractable forms. A more recent example is his reduction of a significant part of William Fulton's scheme-theoretic *Intersection theory* to first-order form, by careful rearrangement of the material.[54] Macintyre's paper illustrates how much useful work in areas related to model theory can be done by concentration and intelligence, with only minimal recourse to model-theoretic devices. (He uses some ultraproducts, but little else.)

The definition of satisfaction

Tarski's truth definition of the 1930s gave, for each structure $\mathcal{M}$ and logic $\mathscr{L}$, a formula $\theta(x)$ of some appropriate form of higher-order logic such that

$$\theta(\varphi) \text{ iff } \varphi \text{ is a sentence of } \mathscr{L} \text{ that is true in } M.$$

[*See §12.A.*] The revised form in his later paper with Vaught gave a formula $\theta(x, y)$ of set theory such that for every structure $\mathcal{M}$ and first-order sentence with symbols appropriate for $\mathcal{M}$,[55]

$$\theta(\mathcal{M}, \varphi) \leftrightarrow \mathcal{M} \text{ is a model of } \varphi. \tag{2}$$

[*See §1.3.*] Both truth definitions used induction on the complexity of formulas, and as a result of this the revised form actually gave a set-theoretic definition of the relation

$$\text{The sequence } \bar{a} \text{ of elements of } \mathcal{M} \text{ satisfies the formula } \varphi.$$

The earlier definition was given for a single fixed structure; the later allowed the structure to vary, but also involved no comparison of structures.

It is rather rare for model theorists to give arguments that refer to the existence of set-theoretic formulas defining truth or satisfaction in structures.[56] On the other hand the recursive clauses of Tarski's truth definition are used constantly, often without explicit mention. For example $\exists x \varphi(x)$ is true in $\mathcal{M}$ if and only if some element of $\mathcal{M}$ satisfies $\varphi(x)$.

Already in 1949 Abraham Robinson gave a recursive definition of a formula θ as in (2), but without invoking the notion of elements satisfying a formula.[57] He was able to do this by adding an assumption that every element of a structure is associated with an individual constant. [*See §1.5, Definition 1.5.*] This association could

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[53] Denef 1984. [54] Fulton (1984) and Macintyre (2000b). [55] Tarski and Vaught (1958).

[56] Such arguments do occur in what Barwise (1972) called 'soft model theory', which deduces model-theoretic theorems from the fact that the formula defining satisfaction is set-theoretically absolute. [*See §9.A.*] [57] A. Robinson (1951: 19–21).

be 'possibly only in passing': if a structure $\mathcal{M}$ has elements with no corresponding individual constant, then new individual constants can be added for purposes of the truth definition. The assumption proved to be a valuable device for mathematical purposes, because it led directly to Robinson's notion of the *diagram* of a structure. The diagram D of $\mathcal{M}$ is the set of all atomic or negated atomic sentences true in $\mathcal{M}$, in a language where every element has a corresponding individual constant. [*See §15.4, footnote 35.*] Then $\mathcal{M}$ is embeddable in $\mathcal{N}$ if and only if $\mathcal{N}$ is a model of D.[58] Likewise, we can take the *complete diagram* of $\mathcal{M}$ to be the set of all first-order sentences true in $\mathcal{M}$ with constants for all elements; then $\mathcal{M}$ is elementarily embeddable in $\mathcal{N}$ if and only if $\mathcal{N}$ is a model of the complete diagram of $\mathcal{M}$. These devices became valuable tools of the paradigm shift which Robinson initiated, to make mappings between structures a central notion of model theory; see §18.5 below.

Robinson's truth definition was serendipity. His original reason for assuming the individual constants was that he learned his logic not from Tarski but from Rudolf Carnap, and Carnap had assumed that each element of a 'state-description'—his nearest counterpart of a structure—was named by a constant.[59] Carnap's involvement in this area was almost as old as Tarski's. In 1932, Gödel wrote to Carnap that he was intending to publish "eine Definition für 'wahr'";[60] this was in the context of arithmetic, where every element is named by a constant term. Gödel never published it, and we can only guess how it would have gone.[61]

Following Mal'tsev,[62] many authors have found it convenient to use the notion of the *signature* of a structure or a language, which is the set of relation, function and individual constant symbols of the language. [*See §1.1, Definition 1.1.*] An earlier notion playing a similar role was the *similarity type*, following McKinsey and Tarski: 'Two algebras [...] are called *similar* if the number of operations is the same in both algebras and if the corresponding operations [...] are operations with the same number of terms'.[63]

18.4 Building a structure

The method of quantifier elimination serves to analyse structures that we already have. But model theory relies also on methods for building new structures with specified properties.

In his paper of 1915 on the calculus of relatives, Löwenheim showed that every sentence of first-order logic, if it has a model, has a model with at most countably many

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[58] Cf. A. Robinson (1956b: 24). [59] See for example Carnap's definition of 'holds in a state-description', Carnap (1947: 9). [60] Gödel (2003: 346–7). [61] See Feferman (1998). [62] Mal'tsev (1962). [63] McKinsey and Tarski (1944: 190).

elements. His proof has several interesting features, including his introduction of function symbols to reduce the satisfiability of a sentence:[64]

$$\forall x \exists y \varphi(x, y)$$

to the satisfiability of the sentence

$$\forall x \varphi(x, F(x)).$$

Thus it seems that Löwenheim invented Skolem functions, if we forgive him his bizarre explanation of the passage from the first sentence to the second. Löwenheim's starting assumption is that a given sentence φ is 'satisfied' in some domain; this means the same as saying that some structure is a model of φ, but Löwenheim never mentions the structure, which is another reason why his proof is hard to follow.[65]

Skolem tidied up Löwenheim's argument and strengthened the result.[66] He showed, using a coherent account of Skolem functions, that if T is a countable first-order theory with a model $\mathcal{M}$, then T has a model $\mathcal{N}$ with at most countably elements. (In fact he allowed countable conjunctions and disjunctions in the sentences of T too, and infinite quantifier strings.) The proof shows that $\mathcal{N}$ can be taken as an elementary substructure of $\mathcal{M}$, but at this date Skolem lacked even the notion of substructure. [*See §4.1 Definition 4.3, §3.8 Definition 3.7.*] Because values have to be chosen for the Skolem functions, and the starting structure need not allow these values to be defined explicitly (for example it may have too many automorphisms), Skolem had to assume the axiom of choice.

Skolem's argument was adapted and generalised in many ways. For example if κ is an infinite cardinal, $\mathscr{L}$ is a signature of cardinality at most κ, and $\mathcal{M}$ is a structure of signature $\mathscr{L}$ containing a set of elements X of cardinality at most κ, then $\mathcal{M}$ has an elementary substructure of cardinality at most κ containing all the elements of X. This is for first-order logic, but most logics allow analogous results. Theorems of this type came to be called *Downward Löwenheim–Skolem Theorems*. [*See §7.3, Theorem 7.2(1).*] Takeuti is said to have joked that Downward must be a very clever person to have so many theorems.

Later, Skolem made an adjustment of his argument which was fateful for model theory.[67] Starting from a structure $\mathcal{M}$, he built a new structure $\mathcal{N}$; but the elements of $\mathcal{N}$ were not elements of $\mathcal{M}$, they were all the ordinals below an ordinal α. (He chose $\alpha = \omega$, so that the elements of $\mathcal{N}$ were natural numbers.) The construction of $\mathcal{N}$ was inductive, with infinitely many steps. At each step a choice was made that ensured that certain elements would satisfy a certain formula. (For example

[64] Löwenheim (1915: ¶4 in the proof of Theorem 2). [65] See the analysis in Badesa (2004).
[66] Skolem (1920). [67] Skolem (1922).

if the formula was $\exists x R(x, y)$ and n was a given natural number, then it might be specified that $R(m, n)$ holds, where m is the first natural number not so far used; the well-ordering of α made this choice well-defined.) Some combinatorics was invoked to ensure that by the end of the construction $\mathcal{N}$ would have all the required properties.[68]

Using this scheme, Skolem showed, without using the axiom of choice, that if T is a countable first-order theory and T has a model, then T has a model with at most countably many elements.[69] It was on this basis that he stated *Skolem's Paradox*: if Zermelo–Fraenkel set theory is consistent then it has a countable model, so that 'There are uncountable cardinals' is satisfied in a countable domain. [*See §8.2.*]

The scheme allows many variations: a larger ordinal can be used, different starting assumptions can be fed in, different combinatorics can be invoked. The earliest variation came in the 1930 doctoral thesis of Gödel.[70] Gödel started not with Skolem's assumption that the theory T has a model, but with the assumption that no contradiction can be deduced from T within a standard proof calculus. In this way Gödel proved *completeness* for first-order logic: if no contradiction can be deduced from the countable first-order theory T, then T has a model. [*See §4.A, Theorem 4.24.*] Using the fact that proofs are finite, he pointed out the consequence that a countable first-order theory has a model if and only if every finite subset of it has a model; this is the *Compactness Theorem* for countable first-order logic. [*See §4.1, Theorem 4.1.*] In fact we can prove the Compactness Theorem without mentioning formal deductions, by moving back halfway to Skolem's construction; instead of assuming, as Skolem did, that T has a model, we assume that every finite subset of T has a model. (This device is not in Gödel's paper, but later it became common knowledge.)

To prove Completeness for uncountable theories in first-order logic, the same scheme works but with an uncountable cardinal in place of ω, and more careful combinatorics to justify the induction. This was done first by Mal'tsev (1936), and later but independently by Leon Henkin and by Abraham Robinson.[71] Probably the version most commonly used today is Henkin's neat second attempt, as filtered through Gisbert Hasenjaeger.[72] Henkin's method prepares the theory before the inductive construction begins. The preparation includes expanding T to a maximal syntactically consistent set—which in general requires the axiom of choice. Again we can convert the proof to a proof of the Compactness Theorem for first-order theories of any cardinality, by the same device as in the previous paragraph. [*See §§4.A–4.B.*]

[68] In Skolem (1922) a finite set of alternative choices were made at each step, creating a tree of choices; then a form of König's tree lemma was invoked to ensure that at least one branch of the tree is infinite and hence meets the requirements. [69] Skolem (1922). [70] Gödel (1931). There is some doubt how far Gödel was aware of Skolem (1922); see van Atten and Kennedy 2009. [71] Henkin (1949) from his PhD thesis of 1947; A. Robinson (1951) from his PhD thesis of 1949. [72] Hasenjaeger (1953).

Another variation of Skolem's scheme is omitting types. The *type* of a tuple $\bar{a}$ of elements in a structure $\mathcal{N}$ is the set $\Phi(\bar{x})$ of all formulas $\varphi(\bar{x})$ such that $\bar{a}$ satisfies $\varphi(\bar{x})$ in $\mathcal{N}$; $\mathcal{N}$ is said to *realise* the types of its tuples of elements. If X is a set of elements of N and the formulas $\varphi(\bar{x})$ are allowed to contain constants for the elements of X, we say that $\Phi(\bar{x})$ is a *type over X*. [*See §14.1.*]

The type of a tuple $\bar{a}$ of elements of $\mathcal{N}$ is an infinite set of formulas. This allows the possibility that the type of $\bar{a}$ is not yet determined at any finite step in the construction of $\mathcal{N}$; so if $\Phi(\bar{x})$ is a particular set of formulas, we have enough opportunities in the construction to ensure that the type of $\bar{a}$ in $\mathcal{N}$ is not $\Phi(\bar{x})$. If $\mathcal{N}$ is countable then there are countably many tuples of elements, and we can interweave the requirements so as to ensure that each of countably many sets $\Phi(\bar{x})$ is *omitted* in $\mathcal{N}$, in the sense that no tuple in $\mathcal{N}$ has $\Phi(\bar{x})$ as its type. (This presupposes that the sets $\Phi(\bar{x})$ are *non-principal*, i.e. not determined by a finite part of themselves.) Each set omitted can be 'over' a finite number of elements of $\mathcal{N}$.

In 1959 Vaught gave the classic omitting types theorem for countable models of complete first-order theories.[73] This theorem allows one to omit countably many types at once; Vaught attributes this feature to Andrzej Ehrenfeucht. The paper also contains *Vaught's Conjecture* as a question: 'Can it be proved, without the use of the continuum hypothesis, that there exists a complete theory having exactly $\aleph_1$ non-isomorphic denumerable models?' (The Conjecture is that there is no such theory. Some special cases of the Conjecture have been proved; at the time of writing it is still unresolved whether a counterexample has been given.)

There were several close variants of omitting types. The Henkin–Orey theorem was one that appeared before Vaught's paper, while Robinson's finite forcing and Grilliot's theorem on constructing families of models with few types in common were two that came later.[74] Martin Ziegler made finite forcing more palatable by recasting it in terms of Banach–Mazur games;[75] the same recasting works for all versions of omitting types.

Finite forcing builds existentially closed models; these were introduced into model theory by Michael Rabin and Per Lindström.[76] During the 1970s Oleg Belegradek, Ziegler, Saharon Shelah and others put a good deal of energy into constructing existentially closed groups, after Macintyre had shown that they have remarkable definability properties.[77]

Skolem's scheme also allows the use of set-theoretic prediction principles. These are set-theoretic statements, some provable in Zermelo–Fraenkel set theory and some true in the constructible universe or merely consistent, which tell us that certain things are guaranteed to happen a large number of times (for example on a

[73] Vaught (1961). [74] Orey (1956), Barwise and A. Robinson (1970), and Grilliot (1972).
[75] Ziegler (1980). [76] Rabin (1964) and Lindström (1964). [77] Macintyre (1972).

stationary subset of an uncountable cardinal). Such principles were first pointed out by Ronald Jensen;[78] Shelah added Jensen's principles and some of his own to the arsenal of model-theoretic techniques.[79] In this work, the boundaries between set theory, model theory, and abelian group theory become very thin.

The Compactness Theorem can often allow us to build structures without having to go through the combinatorics needed to prove the Compactness Theorem itself. For example, given the Compactness Theorem, it is easy to prove that if λ is an infinite cardinal, $\mathscr{L}$ is a signature of cardinality at most λ, L is a first-order language of signature σ, and $\mathcal{M}$ is an infinite structure of signature $\mathscr{L}$ with fewer than λ elements, then $\mathcal{M}$ has an elementary extension of cardinality λ. One takes the complete diagram of $\mathcal{M}$, adds λ new individual constants together with inequations to express that the new constants stand for distinct elements, and then notes that every finite subset of the resulting theory has a model by interpreting the finitely many new constants in $\mathcal{M}$. This result became known as the *Upward Löwenheim–Skolem–Tarski Theorem*—though Tarski's name was generally dropped. [*See §7.3, Theorem 7.2(2).*] The irony was that it was Skolem,[80] not Tarski, who for antiplatonist reasons refused to accept that the theorem was true (though he allowed that it might be deducible within some formal set theories).

The Upward Löwenheim–Skolem Theorem above was first stated by Tarski and Vaught, though the proof above by Compactness was essentially as in Mal'tsev's proof of a weaker result.[81] Tarski had claimed in 1934 that in 1927/8 he had proved that every consistent first-order theory with no finite model has a model with uncountably many elements.[82]

Combinatorics could be added to Compactness to get further results. Ehrenfeucht and Mostowski showed, using Compactness and Ramsey's Theorem, that if T is a complete first-order theory with infinite models and $(X, <)$ is a linearly ordered set, then T has a model $\mathcal{M}$ whose domain includes X, and for each finite n, any two strictly increasing n-tuples from X satisfy the same formulas in $\mathcal{M}$.[83] Thus $(X, <)$ is what later came to be called an *indiscernible sequence* in $\mathcal{M}$. [*See §15.5, Definition 15.20.*] If $\mathcal{M}$ is the closure of X under Skolem functions (as we can always arrange), $\mathcal{M}$ is said to be an *Ehrenfeucht–Mostowski model* of T.

Ehrenfeucht–Mostowski models have tightly controlled properties. For example they realise few types (see their use in §18.7 below). By choosing $(X, <)$ and $(X', <')$ sufficiently different, we can often ensure that the Ehrenfeucht–Mostowski models constructed over these two ordered sets are not isomorphic; this is the basic idea underlying many of Shelah's constructions of large families of nonisomorphic

[78] Jensen (1972). [79] See for example the use of Shelah's 'black box' to construct abelian groups with interesting properties, in Corner and Göbel (1985). [80] Skolem (1955). [81] Tarski and Vaught (1958) and Mal'tsev (1936). [82] The claim is in a note added by the editors to the end of Skolem (1934). Vaught (1954: 160) reports the few facts that are known about this early proof by Tarski. [83] Ehrenfeucht and Mostowski (1956).

models (again see §18.7). One can also construct Ehrenfeucht–Mostowski models of infinitary theories, using various theorems of the Erdős–Rado partition calculus in place of Ramsey's Theorem. As a byproduct we get a versatile way of building *two-cardinal models*, i.e. models of first-order theories in which some definable parts have one infinite cardinality and others have another infinite cardinality, as Michael Morley showed.[84] (Vaught had obtained two-cardinal results earlier by other methods.)

In his doctoral dissertation of 1966 Jack Silver, building on work of Haim Gaifman and Frederick Rowbottom, showed that if the set-theoretic universe contains a measurable cardinal (or even an Erdős cardinal), then the constructible universe forms an Ehrenfeucht–Mostowski model whose indiscernibles are a class of ordinals which includes all uncountable cardinals. Silver's dissertation was published as 'Some applications of model theory in set theory';[85] but the Silver indiscernibles rapidly took on a life of their own as one of the fundamental notions of large cardinal theory.

Other proofs of the Compactness Theorem were found later. Among the most elegant, one was found by Edward Frayne, Anne Morel, and Dana Scott using ultraproducts (on which see §18.6 below),[86] after Tarski had noticed that reduced products can be used to prove Compactness for sets of Horn sentences. [*See §13.C, Corollary 13.23.*] A quirky but extremely neat proof of the Compactness Theorem was found later by Itai Ben-Yaacov, using a fragment of first-order logic called positive logic.[87]

There is another general procedure for building structures; it goes by the name of *interpretation*. [*See Chapter 5.*] We illustrate with the familiar construction of the field $\mathbb{Q}$ of rational numbers from the ring $\mathbb{Z}$ of integers. Suppose $\mathbb{Z}$ is given. We select a definable relation on $\mathbb{Z}$, namely the set of all ordered pairs (m, n) with $n \neq 0$; a formula $\varphi_{\mathrm{dom}}(x, y)$ defines this relation in $\mathbb{Z}$. We define an equivalence relation on these pairs: $(m, n) \sim (m', n')$ if and only if $mn' = m'n$; a formula $\varphi_{\sim}(x, y, x', y')$ defines this relation. The elements of the structure $\mathbb{Q}$ will be the equivalence classes of $\sim$. We define the operation $\times$ on the equivalence classes, by defining it on representatives:

$$(m, n) \times (m', n') = (m'', n'') \text{ iff } mm'n'' = m''nn'.$$

Again this is definable in $\mathbb{Z}$ by a formula $\varphi_{\times}(x, y, x', y', x'', y'')$. Likewise with $+$ and $-$, and $^{-1}$ too if we find a suitable conventional value for 0^{-1}. The outcome is that the instructions for building $\mathbb{Q}$ from $\mathbb{Z}$ are coded up as a bundle Γ of formulas in the language of $\mathbb{Z}$, indexed by the operations of $\mathbb{Q}$ together with formulas defining the

[84] Morley (1965b). [85] Silver (1971). [86] Frayne et al. (1962/1963). [87] Ben-Yaacov (2003).

equivalence classes that form the elements of $\mathbb{Q}$. We can summarise the situation by writing $\mathbb{Q} = \Gamma(\mathbb{Z})$. The bundle Γ is the *interpretation*.

Note that if R is any other integral domain then we can form $\Gamma(R)$ with the same Γ; it will be the field of fractions of R. Note also that if ψ is any sentence in the first-order language of $\mathbb{Q}$, then via Γ there is a sentence ψ^Γ such that ψ^Γ holds in R if and only if ψ holds in $\Gamma(R)$. If ψ^Γ can be effectively calculated from ψ, and the set of sentences true in R is recursive, then it is decidable whether or not ψ holds in $\Gamma(R)$.

Mostowski, Tarski, Mostowski, et al., Mal'tsev, and Ershov gave definitions of the notion of interpretation.[88] To construct the domain of the new structure, Mostowski and Tarski used single elements; Mal'tsev used ordered triples of elements, and Ershov introduced a definable equivalence relation on n-tuples. In the 1970s model theorists became interested in the question what structures are interpretable in a given structure, and Ershov's notion of interpretation was generally the one they used. Shelah described how one might think of the elements of structures interpretable in a structure $\mathcal{M}$ as *imaginary elements* of $\mathcal{M}$.[89]

Hilbert and Bernays noticed that if the theory T, suitably encoded as a set of natural numbers, is definable in the structure $\mathbb{N}$ of natural numbers, then Gödel's completeness proof can be carried out within first-order arithmetic, and the effect is that the built model $\mathcal{N}$ of T has the form $\Gamma(\mathbb{N})$ for an interpretation Γ defined in terms of T.[90] They also put a bound on the arithmetical complexity of the relations of $\mathcal{N}$. This suggestive result points in a number of directions; we mention two.

One direction is to consider structures that are encoded in the natural numbers in such a way that all their relations and functions are recursive. Model theory with the structures taken to be of this form is called *recursive model theory*. Mal'tsev took some early steps in this direction.[91] The textbook of Sergei Goncharov and Ershov could cite nearly 400 references.[92]

Another direction is to exploit the idea of doing model theory within arithmetic, for example constructing models of arithmetic within arithmetic. Ideas akin to this allowed Jeff Paris and Leo Harrington to find, for the first time, a naturally occurring theorem of arithmetic that is provable in set theory but independent of the first-order Peano axioms.[93]

18.5 Maps between structures

During the period 1930–50, mathematicians generally had begun to take a closer interest in the maps between structures. This was the period that saw the invention

[88] Mostowski (1948: 270), Tarski, Mostowski, et al. (1953: 20ff), Mal'tsev (1960a), and Ershov (1974). [89] Shelah (1978: chIII, §6). [90] Hilbert and Bernays (1939). [91] Mal'tsev (1960b). [92] Goncharov and Ershov (1999); see also Ershov et al. (1998). [93] Paris and Harrington (1977).

of category theory. The trend naturally made its way into model theory.

Garrett Birkhoff published his famous characterisation of the classes of models of sets of identities in 1935.[94] Birkhoff's paper uses a number of straightforward model-theoretic facts about mappings, for example that universally quantified equations are preserved under taking homomorphic images; Edward Marczewski extended this fact to all positive first-order sentences and asked for a converse.[95] Tarski reported that his own work on formulas preserved in substructures (the Łoś–Tarski Theorem) was done in 1949–50.[96]

In §18.6 we will examine how these new ideas played out in model theory. In the present section we will see how maps between structures came to play a deeper role in model theory, not just as possible topics but as essential tools of the subject. One can trace this development to two model theorists, Abraham Robinson and Roland Fraïssé. I begin with Robinson.

Abraham Robinson

In his PhD thesis, Robinson considered two algebraically closed fields $\mathcal{M}$ and $\mathcal{N}$ of the same characteristic.[97] By juggling upwards and downwards Löwenheim–Skolem arguments, he found algebraically closed fields $\mathcal{M}^*$ and $\mathcal{N}^*$ which both have transcendence degree ω, such that the same first-order sentences hold in $\mathcal{M}^*$ and $\mathcal{M}$ (so that $\mathcal{M}$ and $\mathcal{M}^*$ have the same characteristic), and the same holds for $\mathcal{N}$ and $\mathcal{N}^*$. Then he quoted Steinitz's Theorem, that two algebraically closed fields of the same characteristic and the same transcendence degree are isomorphic. From this he deduced that the same first-order sentences hold in $\mathcal{M}^*$ and $\mathcal{N}^*$, and hence also in $\mathcal{M}$ and $\mathcal{N}$. So the first-order theory of algebraically closed fields of a given characteristic is a complete theory—it settles all questions in the language.

There were two major novelties here. First, Robinson used a known algebraic fact about maps between structures (Steinitz's Theorem) in order to deduce a model-theoretic conclusion. Second, he used complete diagrams so as to construct elementary embeddings. At this date the use of elementary embeddings was only implicit. [*See §4.1, Definitions 4.3–4.4.*] Tarski defined elementary extensions in 1952/3 (though at that date he called them arithmetical extensions) and published them some years later.[98] Conspicuously, Tarski failed even then to define elementary embeddings; 'elementary imbeddings' [sic] appeared in a paper first published in 1961.[99]

Between Robinson's doing this work and publishing it, Tarski published the completeness of the theory of algebraically closed fields of a given characteristic, which he had discovered by the method of quantifier elimination. So the method of quantifier elimination gave Robinson's result, together with other results that didn't

[94] Birkhoff (1935). [95] Marczewski (1951). [96] Tarski (1954). [97] A. Robinson (1951: 59–60). [98] Tarski and Vaught (1958). [99] Kochen (1961).

obviously yield to Robinson's new methods. The next few years saw Robinson working hard to extend his methods to capture Tarski's results and more besides. To this work we owe the notions of model completeness, model companion, differentially closed field, an amalgamation criterion for quantifier elimination, model-theoretic forcing, and Robinson's joint consistency theorem that gave the Craig Interpolation Theorem.

Vaught was one of the first model theorists to exploit the new methods. For example he pointed out, using essentially Robinson's argument, that any countable theory that is λ-categorical for some infinite λ and has no finite models must be complete; this is *Vaught's Test*.[100] [*See §3.B, Proposition 3.10.*] Robinson wrote appreciatively of Vaught's Test, noting that his own argument could be simplified by taking λ uncountable.[101]

Roland Fraïssé

In 1953/4 Fraïssé published two papers in which he pointed out that certain countable structures are in a sense determined by the families of finite structures embeddable in them.[102] Taking the ordered set of rational numbers as a paradigm, he made two important observations.

(a) We can characterise those classes of finite structures which are of the form

all finite structures embeddable in $\mathcal{M}$

for some countable structure $\mathcal{M}$. (Following Fraïssé I shall call these γ-*classes*—it is not a standard name.)

(b) A γ-class has the amalgamation property if and only if $\mathcal{M}$ can be chosen to be homogeneous, and in this case $\mathcal{M}$ is determined up to isomorphism by the γ-class. (A class **K** has the *amalgamation property* if for all embeddings $e_1 : A \longrightarrow B_1$ and $e_2 : A \longrightarrow B_2$ within **K** there are embeddings $f_1 : B_1 \longrightarrow C$ and $f_2 : B_2 \longrightarrow C$, also within **K**, such that $f_1 \circ e_1 = f_2 \circ e_2$. A is *homogeneous* if every isomorphism between finite substructures of $\mathcal{A}$ extends to an automorphism of $\mathcal{A}$.)

By observation (a), Fraïssé introduced into model theory a kind of Galois theory of structures: it invited one to think of a structure as built up by a pattern of amalgamated extensions of smaller structures. This idea became important in stability theory.

By observation (b), Fraïssé introduced the amalgamation property into model theory (though the name came later). Also he provided a way of building countable structures by assembling a suitable γ-class of finite structures; intuitively, one keeps extending in all possible ways, amalgamating the resulting extensions as one goes.

[100] Vaught (1954). [101] A. Robinson (1956b: 11). [102] Fraïssé (1953, 1954a).

His version of the idea was modest, but it continues to be widely used as a source of ω-categorical structures. Ehud Hrushovski used a version of it to construct his 'new strongly minimal set'.[103] [*See §17.3.*]

In 1956 and 1960 Bjarni Jónsson, who had reviewed Fraïssé's 1953-paper, published two papers removing the limitation to finite and countable structures in Fraïssé's construction of homogeneous structures.[104] The price he had to pay was that the generalised continuum hypothesis was needed at some cardinals. Morley realised almost at once that, thanks to the Compactness Theorem, Jónsson's assumptions on the γ-class are verified if one considers the class of all 'small' subsets of models of a complete theory T and replaces embeddings by partial elementary maps—i.e. elementary maps defined on a subset of a model.[105] One feature of the resulting structures $\mathcal{M}$, at least under suitable conditions on the cardinals involved, was that if X was a set of elements of $\mathcal{M}$, of smaller cardinality than M itself, then every type of T over X would be realised in $\mathcal{M}$. This property of $\mathcal{M}$ was called *saturation* (generalising Vaught's notion of a saturated countable structure).[106]

The Morley–Vaught theory tells us that under suitable set-theoretic assumptions, every structure has a saturated elementary extension. These set-theoretic assumptions were always a stumbling block, and so weak forms of saturation were devised that served the same purposes without special assumptions. For example every structure has an elementary extension that is special.[107] Every countable structure has a recursively saturated elementary extension.[108] For every structure $\mathcal{M}$ and cardinal κ, $\mathcal{M}$ has an elementary extension that is κ-saturated, meaning that every type over fewer than κ elements is realised.

Saunders Mac Lane reports that when his student Morley first brought him the material that led to Morley and Vaught 1962, '[...] I said, in effect: "Mike, applications of the compactness theorem are a dime a dozen. Go do something better".'[109] Mac Lane adds that Morley's Theorem (see §18.7 below) was the fruit of this advice. [*See §17.3, Theorem 17.3.*]

In the 1970s there was some debate about how best to handle the Morley–Vaught γ-class. Gerald Sacks proposed one should think of it as a category with partial elementary maps as morphisms.[110] Shelah went straight to a very large saturated model C (but we never ask exactly how large); in his picture the γ-class is simply the class of all small subsets of the domain of C, and the partial elementary maps are the restrictions of automorphisms of C.[111] Shelah's view prevailed. The structure C was known as the *big model* or (following John Baldwin) the *monster model*. Studying

[103] Hrushovski (1992, 1993). [104] Jónsson (1956, 1960). [105] Vaught had come to similar conclusions independently. They published this in Morley and Vaught 1962. Morley and Vaught used a trick from Skolem 1920, adding relation symbols so that partial elementary maps become embeddings. [106] Vaught (1961). [107] Chang and Keisler (1990: 217). [108] Barwise and Schlipf (1976). [109] Mac Lane (1989). [110] Sacks (1972). [111] Shelah (1978: chI §1).

models of a complete first-order T, one could go to a monster model and restrict oneself to subsets of the domain of this model, and elementary maps between them.

In practice the monster model came to be used in a way that reflected Robinson's approach with complete diagrams. Morley and Vaught speak of Jerome Keisler's '"one element at a time" property'.[112] Keisler himself compared his procedure with the element-at-a-time methods used by Cantor and Hausdorff to build up isomorphisms between densely ordered sets.[113] Briefly, the idea was to define a partial elementary map by starting with a well-ordered listing of elements, say $(a_i : i < \kappa)$, and constructing a corresponding listing $(c_i : i < \kappa)$ by induction on i, so that each c_i realises the same type over $(c_j : j < i)$ as a_i realises over $(a_j : j < i)$. Then the mapping $a_i \mapsto c_i$ is elementary. An initial segment of $(c_i : i < \kappa)$ might be given by the problem in hand, and then κ-saturation was invoked to find the remaining elements. Amalgamations would be built up one element at a time: for example given $Y \supset X$ and an element b, one would amalgamate Y and $X \cup \{b\}$ over X, and speak of extending the type of b over X to a type over Y.

Around 1970 category theory was developing fast. People noted that by going with Shelah rather than with Sacks, the model-theoretic community had opted for the analogue of André Weil's 'universal domain',[114] rather than the more recent category-theoretic language of Grothendieck. But other model theorists kept the category connection alive. Michael Makkai and colleagues did some groundwork,[115] but the categorical approach never came to centre stage. Perhaps model theorists enjoy handling elements and dislike morphisms between theories. Nevertheless we can point to one useful outcome: Daniel Lascar visited Makkai and discussed with him the category of elementary embeddings between models of a complete theory. Lascar's enquiries threw up the idea of *Lascar strong type*,[116] which plays a significant role in the study of simple theories and elsewhere.

In the 1970s Saharon Shelah was looking for suitable abstract settings for work in stability theory for infinitary languages. He called one such setting *abstract elementary classes*.[117] An abstract elementary class is a class of structures of some given signature, together with a relation $\prec$ between structures, satisfying certain axioms. The axioms include a variant of Jońsson's axiom of unions of chains; they don't include joint embedding or amalgamation, though these two axioms are added for many applications. Shelah restored the amalgamation viewpoint with a vengeance:[118] to construct structures of cardinality ω_n from countable pieces, he formed n-dimensional amalgams. Shelah carries a remarkable amount of model

[112] Morley and Vaught (1962). [113] Keisler (1961: footnote on Theorem 2.2), his doctoral dissertation. See Cantor (1895) and Hausdorff (1908). [114] Weil (1946: ch.IX §1). [115] Makkai and Paré (1989). [116] Lascar (1982). [117] Shelah (1987a) and Grossberg (2002). [118] Shelah (1983).

theory over into the setting of abstract elementary classes, considering that the axioms make no reference to any language—in fact the blurb of his 2009a includes the remark that 'Abstract elementary classes provide one way out of the cul de sac of the model theory of infinitary languages which arose from over-concentration on syntactic criteria'. This is partly explained by Shelah's Presentation Theorem, which states that every abstract elementary class can be got by taking the class of all models of some given first-order theory which omit certain types, and then forming reducts to a smaller signature.

Abstract elementary classes turned out to be a suitable setting for various analogues of first-order model theory. For example Zilber, discussing his 'analytic Zariski geometries', used a notion of stability got by considering these geometries within a suitable abstract elementary class.[119] Also work of Hrushovski, Pillay, and Ben-Yaacov led to the notion of a *compact abstract theory*, or *cat* for short, which forms a setting for the model theory of Banach spaces or of Hilbert spaces.[120] The motivations behind cats and abstract elementary classes are different, but there are links.[121]

In 1964 Jan Mycielski noticed that Kaplansky's notion of an algebraically compact abelian group (today more often called a pure-injective abelian group) has a purely model-theoretic characterisation that is a close analogue of saturation.[122] With colleagues in Wrocław, Mycielski developed this observation into a theory of *atomic compact structures*, which was useful on the borderline between model theory and universal algebra.

Since atomic compact structures have a large amount of symmetry, they tend to have neat algebraic structural descriptions too; in fact this was the reason for Kaplansky's interest in them. To some extent the same holds for saturated structures, and even for κ-saturated structures when κ is large enough. For example in 1970 Paul Eklof and Edward Fischer (and independently Gabriel Sabbagh) noted that every ω_1-saturated abelian group is algebraically compact, and so one can read off the results of Wanda Szmielew's quantifier elimination for abelian groups rather easily from Kaplansky's structure theory.[123] Likewise, Ershov used ω_1-saturated Boolean algebras to recover Tarski's quantifier elimination results for Boolean algebras.[124] Clean methods of this kind quickly became standard practice.

18.6 Equivalence and preservation

Tarski tells us that by 1930 he had defined the relation of *elementary equivalence*, in modern symbols: $\mathcal{M} \equiv \mathcal{N}$ if the same first-order sentences are true in $\mathcal{M}$ as in

[119] Zilber (2010: 137). [120] Ben-Yaacov (2003). [121] See Baldwin (2009: 36), and his references there. [122] Mycielski (1964). [123] Eklof and Fischer (1972) and Szmielew (1955). [124] Ershov (1964).

$\mathcal{N}$.[125] [*See §2.4, Definition 2.4.*] But it was only in 1950 that he claimed to have a mathematical (as opposed to metamathematical) definition of this notion.[126] His definition went by cylindrifications and made no reference to sentences or formulas being satisfied in structures. In 1946 he had asked for 'a theory of [elementary] equivalence of algebras as deep as the notions of isomorphism, etc. now in use'.[127]

Model theorists evidently found Tarski's cylindrical definition of $\equiv$ unappealing, and soon two other 'mathematical' characterisations of the notion appeared.

Ultraproducts

In 1955 Jerzy Łoś described a construction based on Cartesian products $\mathcal{M} = \text{Prod}_{i\in I}\mathcal{N}_i$ of structures of some fixed signature $\mathcal{L}$.[128] An ultrafilter D on I (i.e. a maximal filter on the powerset $\wp(I)$) is given. [*See §13.2, Definition 13.3.*] Each relation symbol R of $\mathcal{L}$ is defined to hold of a tuple $\bar{a}$ of elements of $\mathcal{M}$ if and only if the set

$$\{i \in I : R\bar{x} \text{ is satisfied in } \mathcal{N}_i \text{ by the projection of } \bar{a} \text{ at } \mathcal{N}_i\}$$

is in the ultrafilter D; and corresponding clauses hold for function and constant symbols. Equality is read this way too, so that any two elements of the product are identified if and only if the set of indices where they agree is in D. [*See §13.c.*] The resulting structure is called an *ultraproduct* of the $\mathcal{N}_i$, or an *ultrapower* if the $\mathcal{N}_i$ are all equal. Łoś showed that if $\varphi(\bar{x})$ is a first-order formula of signature $\mathcal{L}$, and $\bar{a}$ a tuple of elements of the product, then $\bar{a}$ satisfies $\varphi(\bar{x})$ in the ultraproduct if and only if the set of indices i at which the projection of $\bar{a}$ satisfies $\varphi(\bar{x})$ in N_i is a set in the ultrafilter; this is *Łoś's Theorem.* [*See §13.c, Theorem 13.22.*] Łoś's Theorem was new, but it came to light that ultraproducts or their close relatives had been used earlier by Skolem, Hewitt, and Arrow.[129] Skolem's application was model-theoretic, to build a structure elementarily equivalent to the natural numbers with $+$ and $\times$ but not isomorphic to them.

We remarked in §18.4 above that ultraproducts give a fast and efficient proof of the Compactness Theorem. It can be done in several ways. For example let T be a nonempty first-order theory such that every finite subset of T has a model. Let I be the set of finite subsets of T, and for each $i \in I$ let $\mathcal{N}_i$ be a model of i. For each sentence $\varphi \in T$ let X_φ be the set of finite subsets of T that contain φ. Then all intersections of finitely many sets X_φ are nonempty, so there is an ultrafilter D on I containing each X_φ. It follows at once by Łoś' Theorem that the resulting ultraproduct is a model of T. [*See §13.c, Corollary 13.23.*]

By suitable choice of index set and ultrafilter one can ensure that ultraproducts are κ-saturated, for any required κ. Keisler exploited this fact to show, with the

[125] Tarski (1935b: Appendix). [126] Tarski (1952: 712). [127] Tarski (2000: 27). [128] Łoś (1955b).
[129] Skolem (1931), Hewitt (1948), and Arrow (1950).

help of the generalised continuum hypothesis, that two structures are elementarily equivalent if and only if they have isomorphic ultrapowers.[130] Ten years later Shelah proved the same theorem without assuming the generalised continuum hypothesis, and hence gave a 'purely mathematical' characterisation of elementary equivalence.[131] Kochen gave another characterisation of elementary equivalence, using direct limits of ultrapowers.[132]

Thus it turned out that ultraproducts were useful largely because of their high saturation. Since there are other ways of getting highly saturated models of a theory, this made ultraproducts one of the less essential tools of model theory—though some model theorists keep them on hand as a concrete and transparent construction. There are also a few important theorems for which ultraproducts give the only known reasonable proofs; one is Keisler's theorem that uncountably categorical theories fail to have the finite cover property.[133] But they never achieved in model theory the central role that they came to play in set theory, thanks to Scott.[134]

Back-and-forth equivalence

Fraïssé found another way of characterising elementary equivalence without mentioning formulas. He described a hierarchy of interrelated families of partial isomorphisms between structures.[135] In terms of this hierarchy he gave necessary and sufficient conditions for two relational structures to agree in all prenex first-order sentences with at most n alternations of quantifier, where n is any natural number. So $\mathcal{M} \equiv \mathcal{N}$ if $\mathcal{M}$ and $\mathcal{N}$ agree in this sense for all finite n. Fraïssé's paper was unfortunately hard to read, and his ideas became known through a paper of Ehrenfeucht who recast them in terms of games.[136] Soon afterwards they were rediscovered by the Kazakh mathematician Asan Taimanov.[137]

In Ehrenfeucht's version, two players play a game to compare two structures $\mathcal{M}$ and $\mathcal{N}$. The players alternate; in each step, the first player chooses an element of one structure and the second player then chooses an element of the other structure. The second player loses as soon as the elements chosen from one structure satisfy a quantifier-free formula not satisfied by the corresponding elements from the other structure. (Mention of formulas here is easily eliminated.) This is the *Ehrenfeucht-Fraïssé back-and-forth game* on the two structures. For a first-order language with finitely many relation and individual constant symbols and no function symbols, one could show that $\mathcal{M}$ and $\mathcal{N}$ agree in all sentences of quantifier rank at most k if and only if the second player has a strategy that keeps her alive for at least k steps. [*See §16.6.*] Hence $\mathcal{M}$ is elementarily equivalent to $\mathcal{N}$ if and only if for each finite k, the second player can guarantee not to lose in the first k steps.

[130] Keisler (1961). [131] Shelah (1971a). [132] Kochen (1961). [133] Keisler (1967). [134] Scott (1961). [135] Fraïssé (1956). [136] Ehrenfeucht (1960/1961). [137] Taimanov (1962).

With this equipment it is very easy to show, for example, that if $\mathcal{G}, \mathcal{G}'$ are elementarily equivalent groups and $\mathcal{H}, \mathcal{H}'$ are elementarily equivalent groups, then the product group $\mathcal{G} \times \mathcal{H}$ is elementarily equivalent to $\mathcal{G}' \times \mathcal{H}'$.

The beauty of this idea of Fraïssé and Ehrenfeucht was that nothing tied it to first-order logic. Ehrenfeucht himself used it to prove the equivalence of various ordinal numbers as ordered sets with predicates for $+$ and $\times$, in a language with a second-order quantifier ranging over finite sets.[138] Carol Karp adapted it to infinitary logics,[139] and it reappeared in Chen Chung Chang's construction of Scott sentences.[140] [*See §16.6 Theorem 16.4.*] Today, theoretical computer scientists know it in a thousand different forms.

We turn to applications of all this machinery. One striking application of elementary equivalence was Abraham Robinson's creation of nonstandard analysis in 1961.[141] [*See Chapter 4.*] He used the Compactness Theorem to form an elementary extension $^*\mathbb{R}$ of the field $\mathbb{R}$ of real numbers (with any further relations attached) containing infinitesimal elements. He noted that if a theorem of real analysis can be written as a first-order sentence φ, then to prove φ it suffices to use the infinitesimals to show that φ is true in $^*\mathbb{R}$ (a typical example of what Robinson called a *transfer argument*).

James Ax and Kochen in 1965/6 used the new model-theoretic methods to find a complete set of axioms for the field of p-adic numbers (uniformly for any prime p).[142] Their approach was completely different from the method of quantifier elimination, and it seems likely that any proof by that method would have been hopelessly unwieldy. Instead they considered saturated valued fields of cardinality ω_1. Using algebraic and number-theoretic arguments, Ax and Kochen were able to show that under certain conditions, any two such fields are isomorphic. They then wrote down these conditions as a first-order theory T. Assuming the generalised continuum hypothesis, any two countable models $\mathcal{M}, \mathcal{N}$ of T have saturated elementary extensions of cardinality ω_1, which are isomorphic, so that $\mathcal{M}$ and $\mathcal{N}$ must be elementarily equivalent. This proves the completeness of T (and hence its decidability since the axioms are effectively enumerable); a similar argument using saturated structures shows that T is model-complete, and one more push shows that the theory admits elimination of quantifiers. There are various tricks that one can use to eliminate the generalised continuum hypothesis.

This work of Ax and Kochen, together with very similar but independent work of Yuri Ershov,[143] marked the beginning of a long line of research in the model theory of fields with extra structure (for example with valuations or automorphisms). But it hit the headlines because it gave a proof of an 'almost everywhere' version of a

[138] Ehrenfeucht (1960/1961). [139] Karp (1965). [140] Chang (1968). [141] A. Robinson (1961).
[142] Ax and Kochen (1965a,b, 1966). [143] Ershov (1965).

conjecture of Emil Artin on c_2 fields. Since counterexamples to the full conjecture appeared shortly afterwards, 'almost everywhere' was about as much as one could hope for, short of an explicit list of the exceptions.

A notion different from elementary equivalence, but somewhere in the same ballpark, is as follows. Suppose F is a class of mappings between structures, and $\varphi(\bar{x})$ a formula. We say that F *preserves* $\varphi(\bar{x})$ if the following holds: *whenever* $f : \mathcal{M} \longrightarrow \mathcal{N}$ *is a mapping in* F *and* $\bar{a}$ *is a tuple of elements satisfying* $\varphi(\bar{x})$ *in* $\mathcal{M}$, *then* $f(\bar{a})$ *satisfies* $\varphi(\bar{x})$ *in* $\mathcal{N}$. A *preservation theorem* is a theorem characterising, for some class F of mappings, the class of formulas that are preserved by F. For example the Łoś–Tarski Theorem can be paraphrased as characterising the class of formulas preserved by embeddings between models of a given theory.[144]

Stretching the definition above a little, we say that a formula $\varphi(\bar{x})$ is *preserved in unions of chains* when for every chain $(\mathcal{M}_i : i < \beta)$ of structures with union $\mathcal{M}_\beta$ and every tuple $\bar{a}$ of elements of $\mathcal{M}_0$, if $\bar{a}$ satisfies $\varphi(\bar{x})$ in $\mathcal{M}_i$ for each $i < \beta$ then it also satisfies $\varphi(\bar{x})$ in $\mathcal{M}_\beta$. Chang and Łoś and Suszko showed that a first-order formula $\varphi(\bar{x})$ is preserved in unions of chains if and only if it is logically equivalent to a formula of the form $\forall y_1 \dots \forall y_m \exists z_1 \dots \exists z_n \psi$ where ψ has no quantifiers (such formulas are called $\forall_2$ formulas, or Π_2 formulas).[145] In the case where φ is a sentence (no free variables), the main thing to be proved is that if Θ is the set of all $\forall_2$ sentences θ that are consequences of φ, then every model of Θ is elementarily equivalent to the union of a chain of models of φ (and hence is a model of φ). This can be proved by building up a chain whose even-numbered members form an elementary chain of models of Θ, and whose odd-numbered members are models of φ.

The model-building techniques of the previous section were honed on this and many similar problems. The text of Chang and Keisler, first published in 1973, is a compendium of the main achievements of model theory up to that date.[146]

It was natural to ask how far the results of this section could be generalised to other languages; in the 1950s and 1960s this usually meant languages with infinitary features or generalised quantifiers. When someone had introduced a technique for first-order languages, he or she could move on to testing the same technique on stronger and stronger languages. Often a variant of the technique would still work, but set theoretic assumptions and arguments would begin to appear. An observation of William Hanf helped to organise this area: he noted that for any reasonable language L there is a least cardinal κ (which became known as the *Hanf number* of L) such that if a sentence of L has a model of cardinality at least κ then it has arbitrarily large models.[147] A great deal of work and ingenuity went into finding the Hanf numbers of a range of languages.

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[144] Tarski (1954) and Łoś (1955a). [145] Chang (1959) and Łoś and Suszko (1957). [146] Chang and Keisler (1990). [147] Hanf (1960).

One effect of this trend was that during the period from 1950–70 the centre of gravity of research moved away from first-order languages and towards infinitary languages, bringing a heady dose of set theory into the subject. Allow me two anecdotes. In about 1970 a Polish logician reported that a senior colleague of his had advised him not to publish a textbook on first-order model theory, because the subject was dead. And in 1966 David Park, who had just completed a PhD in first-order model theory with Hartley Rogers at MIT, visited the research group in Oxford and urged us to get out of first-order model theory because it no longer had any interesting questions. (Shortly afterwards he set up in computer science, where he applied back-and-forth methods.)

18.7 Categoricity and classification theory

In 1959, Lars Svenonius showed that among countable structures, the models of ω-categorical theories are precisely those structures whose automorphism group has finitely many orbits of n-element sets, for each finite n.[148] Permutation groups with this property are said to be *oligomorphic*.[149] Other model theorists gave other characterisations of ω-categoricity.[150]

Łoś asked: If T is a complete theory in a countable first-order language, and T is λ-categorical for some uncountable λ, then is T also λ-categorical for every uncountable λ?[151] [*See §17.3.*] With hindsight we can see that this was an extraordinarily fortunate question to have asked in 1955, for two main reasons. The first was that at just this date the tools for starting to answer the question were becoming available. If T is λ-categorical and $\mathcal{M}$, $\mathcal{N}$ are models of T of cardinality λ which are respectively highly saturated and Ehrenfeucht–Mostowski, then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic and we deduce that models of T of cardinality λ have very few types to realise. This is strong information. Thus Łoś's question 'stimulated quite a bit of the work concerning models of arbitrary complete theories'.[152]

Second, Łoś's question was unusual in that it called for a description of *all* the uncountable models of a theory. The answer would involve finding a *structure theorem* to explain how any model of the theory is put together. This pointed in a very different direction from Tarski's 'mutual relations between sentences of formalised theories and mathematical systems in which these sentences hold'.[153] One mark of the change of focus was that expressions like 'uncountably categorical' (i.e. λ-categorical for all uncountable λ) and 'totally categorical' (i.e. λ-categorical for all infinite λ), which originally applied to theories, came to be used chiefly for *models* of those theories. For example Walter Baur wrote of '$\aleph_0$-categorical modules'.[154]

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[148] Svenonius (1959). [149] Cf. Cameron (1990). [150] Notably Ryll-Nardzewski (1959). [151] Łoś (1954). [152] Vaught (1963). [153] Tarski (1954). [154] Baur (1975).

In 1965, Michael Morley answered Łoś's question in the affirmative; this is *Morley's Theorem*.[155] [*See §17.3, Theorem 17.3.*] Amid all the literature of model theory, Morley's paper stands out for its clarity, its elegance and its richness in original ideas. Morley's central innovation was *Morley rank*, which assigns an ordinal rank to each definable relation in any model of a theory T which is λ-categorical for some uncountable λ. (In Morley's presentation the rank was assigned to complete types, but later workers generally used the induced rank on formulas or definable relations.) Morley gave the name *totally transcendental* to theories that assign a Morley rank to all definable relations in their models; the terminology came from transcendental extensions in field theory. Morley conjectured that the Morley rank of any uncountably categorical structure (i.e. the Morley rank of the formula $x = x$) is always finite; this was proved soon afterwards by Baldwin and Zilber independently.[156] (As a special case, the Morley rank of an algebraic set over an algebraically closed field is equal to its Krull dimension and hence is finite.)

Baldwin and Lachlan reworked and strengthened Morley's results.[157] [*See §17.3, Theorem 17.6.*] Building on the unpublished dissertation of William Marsh,[158] they showed that each model of an uncountably categorical theory carries a definable *strongly minimal set* with an abstract dependence relation that defines a dimension for the model. Once the strongly minimal set is given, the rest of the model is assembled around it in a way that is unique up to isomorphism. They also showed that the number of countable models of such a theory, up to isomorphism, is either 1 or ω. [*See §17.2.*]

A few young researchers set to work to extend Morley's result to uncountable first-order languages. One of them was Frederick Rowbottom, who introduced the name 'λ-stable' for theories with at most λ types over sets of λ elements;[159] hence the name *stability theory* for this general area.

In 1969, Saharon Shelah began to publish in stability theory.[160] With his formidable theorem-proving skill, he reshaped the subject almost from the start (and some other model theorists fled from the field rather than compete with him). By 1971 he had proved the uncountable analogue of Morley's Theorem.[161] But more important, he had formulated a plan of action for classifying complete theories.

Ehrenfeucht had already noticed that a theory which defines an infinite linear ordering on n-tuples of elements must have a large number of non-isomorphic models of the same cardinality.[162] Shelah saw this result as marking a division between 'good' theories that have few models of the same cardinality, and 'bad' theories that have many. Shelah's strategy was to hunt for possible bad features that a theory might have (like defining an infinite linear ordering), until the list was so comprehensive that a theory without any of these features is pinned down to the point

155 Morley (1965a). 156 Baldwin (1973) and Zilber (1974). 157 Baldwin and Lachlan (1971).
158 Marsh (1966). 159 Rowbottom (1964). 160 Shelah (1969). 161 Shelah (1974a).
162 Ehrenfeucht (1960/1961).

where we can list all of its models in a structure theorem. [*See §17.2.*] As Shelah once explained it in conversation, the outcome should be to show that whenever **K** is the class of all models of a complete first-order theory, 'if **K** is good, it is very very good, but if **K** is bad it is horrid'. Shelah coined the word *nonstructure* for the horrid case, and he suggested several definitions of nonstructure.[163] In one definition, a nonstructure theorem finds a family of 2^λ models of cardinality λ, none of which is elementarily embeddable in any other. In another definition, a nonstructure theorem finds two nonisomorphic models of cardinality λ that are indistinguishable by strong infinitary languages.

Pursuing this planned dichotomy, Shelah wrote some dozens of papers and one large and famously difficult book.[164] Shelah also wrote a number of papers on analogous dichotomies for infinitary theories or abstract classes of structures.[165] His own name for this area of research was *classification theory*. The name applies at two levels: first-order theories classify structures, and Shelah's theory classifies first-order theories.

Shelah himself sometimes suggested that his main interests lay on the nonstructure side:

> I was attracted to mathematics by its generality, its ability to give information where apparently total chaos prevails, rather than by its ability to give much concrete and exact information where we a priori know a great deal.[166]

We should be careful not to deduce too much from this. Shelah's own work on the 'good' side vastly expanded the range of the new tools introduced by Morley. Also it gradually came to light, again mainly through Shelah's own work, that there is not just one dichotomy between good and bad theories; there are many good/bad dichotomies, and they partition the world of complete first-order theories in a complicated pattern. Generally speaking, each dichotomy is defined by the fact that models of theories on the bad side of it have some combinatorial property.[167]

It seemed at first that a minimal requirement for any good structure theory was that the theory should be *stable*, i.e. λ-stable for some cardinal λ. For stable theories, Shelah introduced a notion of relative dependence called *forking*, which reduced to linear or algebraic dependence in classical structures. In terms of forking he defined a class of types which he called *regular*, which carry a dependence relation that gives a cardinal dimension to the set of elements realising them. By the late 1980s it was becoming clear that much of the resulting machinery still worked in theories that were not stable. For example forking still behaved well in a larger class of theories that Shelah had introduced under the name *simple*.[168]

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[163] Shelah (1985). [164] Shelah (1978); the second edition in 1990 reports the successful completion of the programme for countable first-order theories in 1982. [165] E.g. Shelah (1978). [166] Shelah (1987b: 154). [167] At the time of writing, Gabriel Conant has a web page with a map of the main dichotomies: http://www.forkinganddividing.com. [168] Shelah (1980) and Kim (1998).

In stable theories any complete type is in a certain sense 'definable' by first-order formulas.[169] Shelah showed that the definition can always be taken over a *canonical base* which is a family of 'imaginary' elements of the model. A special case of his construction is André Weil's field of definition of a variety,[170] except that here the field of definition consists of ordinary elements, not imaginary ones. Bruno Poizat explained this in 1985 by showing that algebraically closed fields have *elimination of imaginaries*, in the sense that their genuine elements can stand in for their imaginary ones.[171]

Stable groups turned out to have an unexpectedly large amount of structure, much of which carried over to modules (which are always stable). Poizat created a rich theory of stable groups by generalising ideas from Baldwin and Jan Saxl, Zilber, and Cherlin and Shelah.[172] Poizat's framework allows one to rely on intuitions from algebraic geometry in handling stable groups; for example their behaviour is strongly influenced by their generic elements.

One response to the work of Morley and Shelah was to ask what their classifications meant in concrete mathematical situations. The result was a series of papers determining what structures in various natural classes were categorical, totally transcendental and so forth. The first nontrivial paper of this kind was by Joseph Rosenstein on ω-categorical linear orderings.[173] But certainly the most influential was a paper of Macintyre, where he showed that an infinite field is totally transcendental if and only if it is algebraically closed.[174]

Cherlin and Shelah showed that every superstable skew field is an algebraically closed field.[175] In the course of this and related work, both Zilber and Cherlin independently noticed that a group definable in an uncountably categorical structure has many of the typical features of an algebraic group; in Russia the group theorists Vladimir Remeslennikov and Alexandre Borovik were having similar thoughts. Cherlin conjectured that every totally transcendental simple group is up to isomorphism an algebraic group over an algebraically closed field.[176] This became known as *Cherlin's Conjecture*. It was an invitation to model theorists to blend their techniques with those of the classification of finite simple groups. In 2008 Tuna Altınel, Borovik, and Cherlin published a report on the substantial results achieved.[177]

In the preface to that work, the authors wisely comment:

[...] much of the history of pure model theory, which underwent a revolution beginning in the late sixties, and even (or perhaps, particularly) for those who lived through much of the latter, is not easy to reconstruct in a balanced way.[178]

[169] Shelah (1971b), and independently Lachlan (1972). [170] Weil (1946: 68). [171] Poizat (1985: §16e). [172] Poizat (1985); cf. Baldwin and Saxl (1976), Zilber (1977), and Cherlin and Shelah (1980). [173] Rosenstein (1969). [174] Macintyre (1971). [175] Cherlin and Shelah (1980). [176] Cherlin (1979). [177] Altınel et al. (2008). [178] Altınel et al. (2008: xvii).

This is a warning to readers of the three sections below. These parts of model theory are still on the move. I have recorded events and discoveries as I learned of them at the time, but future historians will be much better placed to distinguish the chassis from the bumper stickers.

18.8 Geometric model theory

Geometric model theory classifies structures in terms of their combinatorial geometries and the groups and fields that are interpretable in the structures. The roots of this theory go back to work of Lachlan, Cherlin, and above all Zilber in stability theory in the 1970s, and for this reason the theory is also known as *geometric stability theory*.[179] But by the early 1990s it emerged that the same ideas sometimes worked well in structures that were by no means stable.

An abstract dependence relation gives rise to a combinatorial geometry—in what follows I say just 'geometry'. In this geometry certain sets of points are closed, i.e. they contain all points dependent on them. Zilber classified geometries into three classes:[180] (a) *trivial* or *degenerate*, where all sets of points are closed; (b) non-trivial locally modular, which are not trivial but if a finite number of points are fixed (i.e. made dependent on the empty set), then the resulting lattice is modular—for brevity this case is often called *modular*; (c) the remainder, known briefly as *non-modular*. Classical examples are: for (a), the dependence relation where an element is dependent only on sets containing it; for (b), linear dependence in a vector space; for (c), algebraic dependence in an algebraically closed field.

This classification made its way into model theory rather indirectly. Zilber was working on a proof that no complete totally categorical theory is finitely axiomatisable. (His first announcement of his proof of this result in 1980 was flawed by a writing-up error which was later repaired.)[181] In work on ω-categorical stable theories, Lachlan had introduced a combinatorial structure which he called a *pseudoplane*.[182] A key step in Zilber's argument was to show that no totally categorical structure contains a definable pseudoplane. From this he deduced that the geometry of the strongly minimal set must be either trivial or modular, and his main result followed in turn from this. Cherlin, on reading Zilber's 1980-paper and seeing the error, went to the classification of finite simple groups and proved directly that the strongly minimal set must be either trivial or modular.[183] This result has a purely group-theoretic formulation. In fact several people discovered it independently, and it became known as the *Cherlin–Mills–Zilber Theorem* in honour of three of them. Zilber's proof, which avoids the error mentioned above, reaches the result without the classification of finite simple groups.

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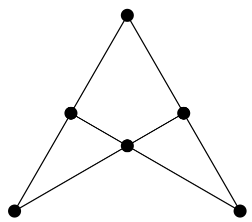
[179] As in the title of Pillay (1996). [180] Zilber (1981). [181] Zilber (1980), and then Zilber (1993). [182] Lachlan (1973/74). [183] Cherlin, Harrington, et al. (1985).

Zilber also called attention to the following combinatorial configuration:[184]



which occurs in modular strongly minimal sets. (The blobs are points of the geometry. All points are pairwise independent. A line between three points means they form a dependent set.) He showed how to construct a group from the configuration; but since this was in the middle of an argument by reductio ad absurdum and quite strong assumptions were in force, it was less than the definitive result. Hrushovski looked closer and showed, using Zilber's configuration, that every modular regular type has an infinite group interpretable in it (in a generalised sense).[185]

When Baldwin and Lachlan had shown that every uncountably categorical structure consists of a strongly minimal set D and other elements attached around it, they found they needed to say something about the way these other elements are attached.[186] Because of categoricity, something in the theory has to prevent the set of attached elements being larger than D. The simplest guess would be that each attached element has to satisfy an algebraic formula (i.e. one satisfied by only finitely many elements) with parameters in D. Baldwin and Lachlan finished their paper with a complicated example to show that this need not hold. Later Baldwin realised that an easy example was already to hand: a direct sum $\mathcal{G}$ of countably many cyclic groups of order p^2 for a prime p. The socle (the set of elements of order at most p) is strongly minimal, in fact a vector space over the p-element field. An element a of order p^2 is described by saying what pa is; but if b is any element of the socle then some automorphism of G fixes the socle pointwise and takes a to $a + b$. In fact the orbit of a over the socle is parametrised by elements of the socle. This parametrisation keeps the orbit from having cardinality greater than that of the socle.

Zilber realised that this was a common pattern in uncountably categorical structures.[187] Each such structure is a finite tower; at the bottom is a strongly minimal set, and as we go up the tower, the orbit of an element over the preceding level in the tower is always parametrised by some group interpretable in that preceding level. He called these groups *binding groups*. There are some cohomological

[184] Zilber (1984a: Lemma 3.3). [185] Hrushovski (1987). [186] Baldwin and Lachlan (1971).
[187] Zilber (1993).

constraints, which allowed Ahlbrandt and Ziegler to begin cataloguing the possibilities.[188] Cherlin and Hrushovski, drawing on these ideas of Zilber and work of Lachlan, proved deep classification results on families of finite structures.[189]

In the light of Zilber's work on uncountable categoricity and its extension by Cherlin, Harrington, and Lachlan,[190] model theorists looked to see what other structures might have modular geometries. One particularly influential result was proved independently by Hrushovski and Pillay, and published jointly:[191] a group $\mathcal{G}$ is modular (i.e. has only modular or trivial geometries) if and only if for each finite n, all definable subsets of G^n are Boolean combinations of cosets of definable subgroups.

We saw that Zilber first applied his trichotomy of geometries by showing that in the particular structures he was considering, the non-modular case never occurred. Zilber now proposed to apply the same trichotomy to another question, namely whether every simple group interpretable in an uncountably categorical structure must be an algebraic group over an algebraically closed field. (Cf. Cherlin's Conjecture above.) Algebraically closed fields themselves have non-modular geometry; at the 1984 International Congress Zilber conjectured the converse, viz. that any uncountably categorical structure with non-modular geometry must be—up to interpretability both ways—an algebraically closed field.[192] This was known as *Zilber's Conjecture*. [*See §17.3.*]

A word about Zilber's motivation may be in order. Macintyre said in 1988 that 'Purely logical classification[s] give only the most superficial general information' (and attributed the point to Georg Kreisel).[193] Zilber was convinced that the opposite must be true: if classical mathematics rightly recognises certain structures as 'good', then it should be possible to say in purely model-theoretic terms what makes these structures good. In fact Zilber in conversation quoted Macintyre (1971) as an example of how a purely model-theoretic condition (total transcendence) can be a criterion for an algebraic property (algebraic closure). Zilber was also convinced that being a model of an uncountably categorical countable first-order theory is an extremely strong property with rich mathematical consequences, among them strong homogeneity and the existence of a definable dimension.

In 1988 Hrushovski refuted Zilber's Conjecture using an ingenious variant of Fraïssé's construction from §18.4 above.[194] But for both Zilber and Hrushovski this meant only that the right condition hadn't yet been found. Since it seemed to be particularly hard to recover the Zariski topology from purely model-theoretic data, a possible next step was to axiomatise the Zariski topology. This is not straightforward: it has to be done in all finite dimensions simultaneously, since the closed sets in dimension n don't determine those in dimension $n + 1$. But Hrushovski de-

[188] Ahlbrandt and Ziegler (1991). [189] Cherlin and Hrushovski (2003). [190] Cherlin, Harrington, et al. (1985). [191] Hrushovski and Pillay (1987). [192] Zilber (1984b). [193] Macintyre (1989). [194] Hrushovski (1993).

scribed a set of axioms, and Zilber and Hrushovski found that, by putting together what they knew, they could prove that Zilber's Conjecture holds for models of the axioms.[195]

Hrushovski proved, for the first time, the geometric Mordell–Lang Conjecture in all characteristics.[196] Key ingredients of his argument were the results on the Zariski topology and on weakly normal groups, and earlier results on the stability of separably closed fields and differentiably closed fields. Hrushovski went on to apply a similar treatment to the Manin–Mumford Conjecture.[197] This case was a little different: the structures in question were unstable. But Hrushovski showed that they inherited enough stability from a surrounding algebraically closed field; and in any case they were 'simple' in Shelah's classification.

18.9 Other languages

In 1885 Charles Peirce, fresh from inventing quantifiers, mentioned that the universal and the existential quantifier are not the only examples.[198] He gave the example of the quantifier 'For two-thirds of all *x*'. Unfortunately, nobody picked up Peirce's idea, until Mostowksi called attention to the quantifiers 'For at least $\aleph_\alpha$ *x*'.[199] Mostowski's paper was timely, because it was useful to have in the 1960s a variety of extensions of first-order logic for testing out new constructions. [*See Chapter 16.*]

Lindström 1969 was another timely paper, in which he gave model-theoretic necessary and sufficient conditions for a logic to have the same expressive power as first-order logic. His result suggested that it might be possible to fit the various logics studied during the previous decade into some higher organisation of logics, within a *generalised* (or *abstract*) *model theory*. Alas, the facts weren't there to support such a theory. The 1970s saw some valiant efforts in this direction, and by the mid-1980s a large amount was known about many different logics extending first-order logic.[200] But the most quotable outcome was that very few logics apart from first-order logic satisfy a Craig Interpolation Theorem.

The mathematical logicians within computer science shrugged their shoulders and asked what is the interest of a logic in which it is impossible to express everyday notions like connectedness, even on finite structures. Thus, for example, Yuri Gurevich:

> The question arises how good is first-order logic in handling finite structures. It was not designed to deal exclusively with finite structures. [...] One would like to enrich first-order logic so that the enriched logic fits better the case of finite structures.[201]

[195] Hrushovski and Zilber (1996). For details see Zilber (2010: Appendix B.2). [196] Hrushovski (1996). [197] Hrushovski (2001). [198] Peirce (1885). [199] Mostowski (1957). [200] See Barwise and Feferman (1985). [201] Gurevich (1984).

One solution was first-order logic with a fixed-point operator added, as proposed by Ashok Chandra and David Harel.[202] The model theory of this logic and its relatives were studied mostly by computer scientists, but this seems to be purely an accident of history; these languages would have been good to have available in the 1960s.

Also of interest to computer scientists were languages with only a finite number of variables. Michael Mortimer launched the topic by showing that in a signature with no function symbols, any consistent first-order sentence using at most two variables has a finite model.[203] Barwise and, independently, Neil Immerman showed how to modify Ehrenfeucht-Fraïssé games to languages with at most n variables;[204] Immerman called the result *pebble games*.

Tarski, in his truth definitions, had taken universal and existential quantification to be dual to each other. This was at variance with a tradition running from Aristotelian logic up to modern formal linguistics, according to which existential quantifiers should be read as disguised Skolem functions.[205] Henkin and Jaakko Hintikka (with the collaboration of Gabriel Sandu) brought this tradition into model theory.[206] These authors noted that by suppressing some of the arguments of the Skolem functions we can increase the expressive power of the language, so as to express independence of one variable from another (as in Henkin's *branching quantifiers*).[207] They also noted that the Skolem functions can be read as strategies for the player $\exists$ in a semantic game between players $\forall$ and $\exists$ that can be used to give a truth definition for sentences; the suppressed arguments correspond to places where the information available to $\exists$ in the game is imperfect.

For model theory a difficulty was that the Skolem function approach to existential quantifiers made it impossible to give a sensible interpretation of subformulas within the scope of an existential quantifier. This problem was resolved by Hodges,[208] who replaced the notion 'tuple $\bar{a}$ satisfies $\varphi(\bar{x})$ in $\mathcal{M}$' by the notion 'the set $\bar{a}, \bar{b}, \ldots$ of tuples satisfies $\varphi(\bar{x})$ in $\mathcal{M}$', thus introducing what Väänänen later called *team semantics*. Peter van Emde Boas noticed that some of the conditions in Hodges' truth definition were identical to conditions appearing in the study of database dependencies. This point was taken up by Väänänen and his colleagues in Helsinki, to develop a model theory of teams, with hopes of using it to bring logic to bear on questions in database theory, statistics, and quantum theory.[209]

[202] Chandra and Harel (1980). [203] Mortimer (1975). [204] Barwise (1977) and Immerman (1982). [205] See the references in Hodges (2015). [206] Henkin (1961) and Hintikka (1996). [207] See also Blass and Gurevich (1986). [208] Hodges (1997b). [209] Abramsky et al. (2016).

18.10 Model theory within mathematics

In their addresses to the 1950 International Congress of Mathematicians at Harvard and MIT, both Abraham Robinson and Tarski expressed the hope that the new subject of model theory—for which neither of them had a name yet—would contribute to algebra and beyond:

> [...] contemporary symbolic logic can produce useful tools—though by no means omnipotent ones—for the development of actual mathematics, more particularly for the development of algebra and, it would appear, algebraic geometry.[210]

> [Model theory has applications] which may be of general interest to mathematicians and especially to algebraists; in some of these applications the notions of [model theory] itself are not involved at all.[211]

Compare these remarks with the comment of Ludwig Faddeev, an observer on the sidelines, in the closing ceremony of the 2002 International Congress of Mathematicians at Beijing:

> Take for instance the sections of logic, number theory and algebra. The general underlining mathematical structures as well as language, used by speakers, were essentially identical.[212]

Job done, one might well say!

Model theory had grown fast. Already the Omega Group bibliography of model theory in 1987 ran to 617 pages.[213] By the mid-1980s there were too many dialects of mathematical model theory for anybody to be expert in more than a fraction. For example, very few model theorists could claim to understand both the work of Zilber and Hrushovski at the edge of algebraic geometry, and the studies by Immerman, Dawar, and other theoretical computer scientists on definable classes of finite structures.

Right from the beginning, model theorists found themselves engaging with other areas of mathematics. In the period from 1950 –70 most of these interactions were with set theory, not with algebra or number theory. From around 1970 there was less interaction with axiomatic set theory. But recent years have seen an increasing amount of discussion between model theory and descriptive set theory. For example when it was realised that Fraïssé's construction in §18.5 above had already been applied by Pavel Urysohn to finite metric spaces with rational distances, the way was open to apply ideas of topological dynamics to Fraïssé-type constructions, as in Alexander Kechris, Vladimir Pestov, and Stevo Todorcevic.[214]

[210] A. Robinson (1952: 694). [211] Tarski (1952: 717). [212] Li (2002: 35). [213] G. H. Müller et al. (1987). [214] Urysohn (1927) and Kechris et al. (2005).

The title of Kechris et al.'s paper mentions Ramsey's Theorem, a reminder that this combinatorial theorem was used in the 1950s to construct Ehrenfeucht–Mostowski models. Links between model theory and combinatorics never ceased, as witness the paper of Maryanthe Malliaris and Shelah relating Szemerédi's Regularity Lemma to the structure theory of graphs stable in the model-theoretic sense.[215] Shelah's book had a twenty-page Appendix of 'the combinatorial theorems needed in the book'.[216]

We have described above some of the interactions between model theory, algebraic geometry and number theory, mostly before the year 2000. More recent years have seen dramatic advances in this area, resting on the earlier work. One example is the *Pila–Wilkie Theorem*, which applies o-minimality in order to bound the numbers of rational points in various sets definable in the real numbers,[217] building on earlier work of Wilkie with o-minimal structures.

Another advance, also closely tied to earlier notions, is the work of Zlil Sela which gives a positive answer to Tarski's question whether all nonabelian finitely generated free groups are elementarily equivalent.[218] The proof ran through several papers and involved building an analogue of diophantine geometry for such groups.[219]

These examples can serve as an indication that future historians of model theory will have plenty of high quality material to write about.

18.11 Notes

Several model theory texts give more detailed historical information about particular theorems; for example Chang and Keisler 1990, Hodges 1993, and Pillay 1996. Dawson 1993 and Lascar 1998 both overlap the present essay. There are surveys on the model-theoretic work of Skolem by Hao Wang (Skolem 1970, 17–52) and on that of Tarski in Vaught 1986.

18.12 Acknowledgments

This essay was originally written in the 1990s at the invitation of Dirk van Dalen for a projected volume on the history of mathematical logic, which sadly never appeared. It was good that Tim Button and Sean Walsh offered a congenial home for it. For their volume I made some corrections and brought the perspective more up to date, but otherwise the content is unaltered.

[215] Malliaris and Shelah (2014). [216] Shelah (1978). [217] Pila and Wilkie (2006). [218] Sela (2006). [219] See also Kharlampovich and Myasnikov (2006) for an alternative approach.

Everybody I ever encountered in model theory should be thanked for their implicit contributions. But I tried to keep a note of those people who helped with specific points in it, and the list is as follows: Zofia Adamowicz, Bektur Baizhanov, John Baldwin, Oleg Belegradek, Tim Button, Greg Cherlin, John W. Dawson, John Doner, Yuri Ershov, Solomon Feferman, Ivor Grattan-Guinness, Marcel Guillaume, Angus Macintyre, Dugald Macpherson, Maria Panteki, Anand Pillay, Gabriel Sabbagh, Hourya Sinaceur, Jouko Väänänen, Robert Vaught, Jan Woleński, Carol Wood, Boris Zilber, Jan Zygmunt. Very probably other people have slipped through the net—my apologies to them.